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Kronecker products for SO(2p) representations

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Abstract. A relatively simple algorithm for the decomposition of the product of two SO(2p) representations is presented. For this purpose, generalised Young tableaux are introduced and their product defined.

1. Introduction

There is today a renewal of interest by orthogonal groups in particle physics in the context of grand unified theories (for a review see, for example, Barbieri 1980, Ellis 1980, Nanopoulos 1980). Among the appealing features that the SO(n) groups present, let us mention the absence of anomaly in their representations, and the property of SO($4n+2$) groups to possess complex conjugate representations. A popular model is given by the SO(10) group which contains a 16-dimensional spinorial representation fitting exactly 16 left-handed elementary fermions of one family. Moreover, models built from SO(n), $n>10$ groups have also been proposed in order to encompass more than one family of fermions (Gell-Mann 1980, Gell-Mann *et al* 1978). For the construction of grand unified models, one needs to know very carefully the mathematical properties of SO(n) Lie algebras as well as of their representations. In particular, the knowledge of the decomposition into irreducible representations of the Kronecker product of two SO(n) representations, which is in itself of mathematical interest, is very useful for the gauge model builder. The reduction of the Kronecker product of representations of O(n) groups has been studied by King (1975a, b) who simplified and generalised the pioneering work of Murnaghan (1938) and Littlewood (1950) based on the character theory and Schur functions. The formulae so obtained (King 1975a, b) are simple and amenable for practical calculations. Unfortunately, they cannot be used for the Kronecker product of SO(2n) representations since an irreducible representation (IR) of O(2n) may split into two irreducible ones under the restriction to SO(2n). This is not the case for orthogonal groups of odd order, the IR of O($2n+1$) being the IR of SO($2n+1$).

With the introduction of difference characters (Murnaghan 1938), Butler and Wybourne (1969) were able to make an extension of Littlewood's results to the SO(2n) case, at the price of rather complicated algorithms. During the completion of our work we received a preprint by Dehuai *et al* (1981) in which the results obtained in Butler and Wybourne (1969) and King (1975a, b) are presented in a systematic way. Let us

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mention another attempt to solve this problem recently proposed by Fischler (1980) based on Young tableau methods for classical groups: unlike for SU(n) groups, this method is not straightforward and generally leads to ambiguous results. In any case, the solutions proposed up to now for analysing the Kronecker product of SO(n) representations require either the introduction of unusual objects (for the physicist) like Schur functions or complicated ways of handling usual objects like Young tableaux.

The method we propose hereafter is obtained in a rather different way and seems to us simpler to use. Actually our method reduces to calculating, given two SO($2n$) IR, a finite number of products of generalised Young tableaux (GYT). As will be defined, a GYT is a tableau which can include 'negative' boxes. The product of two GYT can be seen as a natural extension of the usual product of two SU(n) Young tableaux. Another nice feature of this technique lies in the fact that the rules for products involving vector or spinor representations are essentially the same.

2. Representations of the O(n) and SO(n) groups

The Lie algebra of the O(n) group can be realised with the help of $n(n-1)/2$ infinitesimal generators $J_{ij}=-J_{ji}$, $J_{ij}=J^{\dagger}_{ij}$ which satisfy the commutation relations

$$[J_{ij}, J_{kl}]=\mathrm{i}(\delta_{il}J_{kj}+\delta_{jk}J_{li}+\delta_{ik}J_{jl}+\delta_{jl}J_{ik}). \tag{2.1}$$

The Cartan (maximal Abelian) subalgebra can be chosen as generated by the p commuting generators $J_{12}, J_{2j-1,2j}, \ldots, J_{2p-1,2p}$ if $n=2p$ or $n=2p+1$, the eigenvalues of which, in a given irreducible representation, will yield the weight components.

Any irreducible representation (IR) of the O(n) covering group can be labelled by the components of its greatest weight, i.e. by a set of p, if $n=2p$ or $n=2p+1$, positive numbers m_i, $i=1, 2, \ldots, p$, satisfying $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_p \geqslant 0$. For a given representation, these numbers are integers (true representations) or all half integers (spin representations).

Let us now remind ourselves that any irreducible O($2p+1$) representation is irreducible under SO($2p+1$). This is also the case for an O($2p$) representation with the last component of its greatest weight vanishing ($m_p=0$). The situation is rather different if the last index m_p is non-vanishing. Indeed, in this last case, the O($2p$) representation labelled by $(m_1, \ldots, m_p)$ splits into two IR of SO($2p$) with the corresponding greatest weight $(m_1, \ldots, m_p)$ and $(m_1, \ldots, -m_p)$: two such representations are called conjugate.

Another way of labelling an SO(n) representation, which appears naturally in Cartan's construction, is often used. An SO(n) representation is then characterised by p non-negative integers related to the m_i by the relations

$$q_j=m_j-m_{j+1} \qquad j=1, 2, \ldots, p-1 \qquad \text{if } n=2p$$
$$q_p=m_{p-1}+m_p$$

and

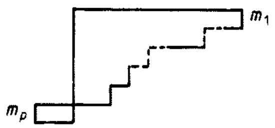
$$q_j=m_j-m_{j+1} \qquad j=1, 2, \ldots, p-1 \qquad \text{if } n=2p+1.$$
$$q_p=2m_p.$$

Let us conclude this short section by recalling that the SO($4\nu+2$) representations $(m_1, \ldots, m_{2\nu+1})$ with $m_{2\nu+1}\neq 0$ are the only SO(n) representations which are complex—actually $(m_1, \ldots, m_{2\nu+1})$ is complex conjugate to $(m_1, \ldots, -m_{2\nu+1})$, all the SO$(4\nu+2)$ representations with $m_{2\nu+1} = 0$, as well as all the representations of SO(4ν) or SO$(2p+1)$, being real or quaternionic real.

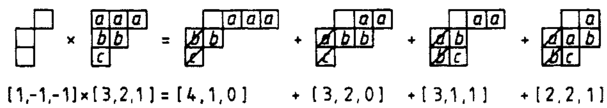
3. Generalised Young tableaux and their product

With the O(n) IR $(m_1, \ldots, m_p)$ $(n = 2p$ or $n = 2p+1)$ it is possible to associate a Young diagram with m_i boxes in the ith row, $i = 1, 2, \ldots, p$ if the IR considered is a true one, or with $m_i - \frac{1}{2}$ boxes in the ith row if the IR is a spinorial one.

What about an SO$(2p)$ IR? If the last index $m_p \geqslant 0$ there is of course no problem, but with its conjugate representation, if $m_p \neq 0$, we shall associate a new type of Young tableau, the last row of which being called a 'negative row'.



Such a tableau will be called the generalised Young tableau or GYT associated with the SO$(2p)$ representation $(m_1, \ldots, m_p)$. In the algorithm we shall propose, we are also led to define a wider class of GYT, i.e. in the context of SO$(2p)$ groups, tableaux associated with the ordered set of positive, null and negative integers $[\alpha_1, \ldots, \alpha_p]$ satisfying $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_p$, the introduction of such tableaux being related to the study of the different weights in an SO$(2p)$ IR. In the following, we shall denote such a GYT by $[, \ldots ,]$. Let us insist on the fact that we can only associate an SO$(2p)$ representation with such a GYT if $\alpha_1 \geqslant \ldots \geqslant \alpha_{p-1} \geqslant |\alpha_p| \geqslant 0$. As an example, in SO(8) the GYT $[1, 0, -1, -2]$ will be represented as follows.



Finally, let us point out that an SO$(2p)$ GYT cannot have more than p rows.

Now let us define the product of two GYT. For simplicity let us distinguish three kinds of products.

(1) A completely arbitrary GYT $[\alpha]$, $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_p$ multiplying a positive GYT $[m]$, $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_p \geqslant 0$.

(2) A completely arbitrary GYT multiplying a negative GYT $[m]$, $0 \geqslant m_1 \geqslant m_2 \geqslant \ldots \geqslant m_p$.

(3) The product of two arbitrary GYT.

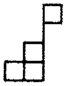
If the product we have to consider concerns two tableaux with only positive rows (classical Young tableaux), then the product law will be the product law of two SU(p) Young tableaux, the only difference being that an SO$(2p)$ GYT with p 'positive rows' $[m_1, \ldots, m_p]$ is *not* equivalent to the simplified one $[m_1 - m_p, \ldots, m_{p-1} - m_p, 0]$.

The multiplication law we define is a direct generalisation of the product of SU(n) Young tableaux (Lichtenberg 1970).

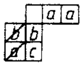
Case 1

Call the boxes in the first line a, those of the second line b and so on up to the pth line of the GYT $[m_1, \ldots, m_p]$ satisfying $m_1 \geqslant \ldots \geqslant m_{p-1} \geqslant m_p \geqslant 0$. Add to the other GYT $[\alpha_1, \ldots, \alpha_p]$ with $\alpha_1 \geqslant \ldots \geqslant \alpha_p$ one box $\boxed{a}$ of $[m]$ using all different ways so that one always gets a GYT. Note that the box $\boxed{a}$ added to the negative row of $[\alpha]$ will cancel the box furthest left in this row. Then add a second $\boxed{a}$ to the obtained tableaux and so on using the usual SU(n) prescriptions..

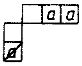
As an illustration, let us consider the following product relative to SO(6).

 = [1, 0, -1, -2]

Notice that the tableau



does not exist since, before adding the b boxes, we would get the tableau

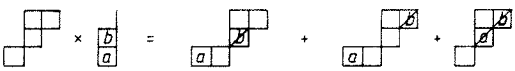


i.e. $[3, -1, 0]$ which is not a GYT.

Case 2

Call the boxes of the *last* line a, those of the line just above b and so on up to the first line of GYT $[m]$. Add to the GYT $[\alpha]$ one box $\boxed{a}$ of $[m]$ in all the possible ways giving always a GYT. Then add a second $\boxed{a}$ and so on using the usual SU(n) prescriptions, but read *from left to right and from down to up*, to satisfy $n_i(a) \geqslant n_i(b) \ldots$ (instead of counting from right to left and from up to down).

As an example



Case 3

The product of two arbitrary GYT can be done using the rules of cases 1 and 2 and the following recurrent formula ($m_i \geqslant 0$, $i = 1, 2, \ldots, k$; $m_j < 0$, $j = k+1, \ldots, p$)

$$[\alpha] \times [m] = ([\alpha] \times [m]_-) \times [m]_+ - [\alpha] \times \{[m]\} \tag{3.1}$$

where

$$[m]_- = [0, \ldots, 0, m_{k+1}, \ldots, m_p]$$

$$[m]_+ = [m_1, \ldots, m_k, 0, \ldots, 0]$$

and $\{[m]\}$ denotes the set of GYT obtained from $[m]$ cancelling in all the possible ways one or more negative boxes with one or more positive boxes ('contraction') with the prescription that two symmetrical, i.e. boxes of the same row, (antisymmetrical, i.e. boxes of the same column) cannot be cancelled by two antisymmetrical (symmetrical) positive boxes.

Example:

$\{[2, 2, -1, -2]\} = [2, 1, 0, -2] + [2, 1, -1, -1] + [1, 1, 0, -1] + [1, 0, 0, 0] + [2, 0, 0, -1]$.

Each element of $\{[m]\}$ has again to be decomposed into a positive and a negative part. It is clear that the decomposition of $[\alpha] \times [m]$ will be obtained by repeating formula (3.1) a finite number of times.

4. Rules for the product of two SO(2*p*) representations

In the analysis of the Kronecker product of two SO(2*p*) IR we need to consider a subset of GYT of crucial usefulness. These are GYT with null and/or negative p labels, say

$$[0, 0, \ldots, 0]; \qquad [0, 0, \ldots, -1, -1]; \qquad [0, 0, \ldots, -2, -2];$$
$$[0, \ldots, 0, -1, -1, -1, -1].$$

We shall denote them in the following way:

$$L_p^{2k}(\{\alpha_i\}) = [0, \ldots, -\alpha_i, -\alpha_i, \ldots, -\alpha_2, -\alpha_2, -\alpha_1, -\alpha_1] \tag{4.1}$$

where α_i is a set of non-negative integers which satisfy $\Sigma_i \alpha_i = k$, $\alpha_i \geq \alpha_{i+1}$. The lower index p just reminds us of the total number of allowed labels, given the rank of the group under consideration. As an example in SO(8) we have

$$L_4^2(1) = [0, 0, -1, -1] \qquad L_4^4(2) = [0, 0, -2, -2] \qquad L_4^4(11) = [-1, -1, -1, -1].$$

Let us now consider the product of two IR of SO(2*p*), $[m_1, \ldots, m_p]$ and $[n_1, \ldots, n_p]$. If m_p and n_p are both negative, it is simpler to make the product $[m_1, \ldots, |m_p|] \otimes [n_1, \ldots, |n_p|]$ and then replace each term of the result by its conjugate. Also, as noted in the preceding section, there are two types of representations of the orthogonal groups: the true representations (T) and the spin representation (S); then there are three types of products to be considered:

$$T_a \otimes T_b = \bigoplus_{i=1}^{I} T_i \qquad S_a \times S_b = \bigoplus_{i=1}^{I} T_i \qquad T_a \otimes S_b = \bigoplus_{i=1}^{I} S_i.$$

With an S-type representation, say $[m_1, \ldots, m_p]$, where m_i are half integers, one associates a tableau $[\mu, \ldots, \mu_p]$ with $\mu_i = m_i - \frac{1}{2}$. Given this procedure, in the case $T_a \otimes S_b$ one must add $\frac{1}{2}$ to each label of each term in the product decomposition, whereas in the case $S_a \otimes S_b$ one adds one unit in the same way to obtain the final result.

Apart from this proviso the three types of product can be made with the same rules.

The general formula can be written in the following compact form for the product of IR $[m] \otimes [n]$ of SO(2*p*)

$$\begin{aligned}
&[\mu] \otimes [\nu] = \Sigma_1 + \Sigma_2 + \Sigma_3 \\
&\Sigma_1 = \sum_{k=0}^{a-1} (L_p^{2k} \times [\nu])_{\mathrm{A}} \times [\mu] \\
&\Sigma_2 = \sum_{k=a}^{a+b-1} \{(L^{2k} \times [\nu])_{\mathrm{A}} \times [\mu] - (L^{2k} \times [\mu])_{\mathrm{NA}} \times [\nu]\} \\
&\Sigma_3 = \sum_{k=a+b}^{Q} \{(L^{2k} \times [\nu])_{\mathrm{A}} \times [\mu] - (L^{2a} \times [\mu])_{\mathrm{NA}} \times (L^{2(k-a)} \times [\nu])_{\mathrm{A}}\}
\end{aligned} \tag{4.2}$$

where a, b are the smallest integers such that $(L^{2a}\times[\mu])$ and $(L^{2b}\times[\nu])$ gives a 'not allowed' GYT (see definition below) and Q is defined by

$$Q=\sum_{i=1}^{2q} n_i \qquad \text{for SO}(4q) \text{ and SO}(4q+2).$$

L^{2k} stands for $L_p^{2k}(\{\alpha_i\})$ with the different possible $\{\alpha_i\}$ as defined in (4.1). Only L^{2Q} is unique and stands for

$$L_p^{2Q}(\ldots,-n_3-n_4,-n_3-n_4,-n_1-n_2,-n_1-n_2).$$

The integers μ_i and ν_i are defined by $\mu_i=m_i$ and $\nu_i=n_i$ (if $[m]$, $[n]$ are of T-type), $\mu_i=m_i-\frac{1}{2}$, $\nu_i=n_i-\frac{1}{2}$ (if $[m]$, $[n]$ are of S-type). It is convenient to choose $[n]$ such that $\Sigma_{i=1}^{p} n_i=N\leqslant M=\Sigma_{i=1}^{p} m_i$ or if $M=N$ such that $n_p\leqslant m_p$; if $m_p=n_p$, $n_{p-1}\leqslant m_{p-1}$ and so on. The subscripts A and NA on the brackets mean that in the product of the GYT one only keeps the terms which fulfil some conditions fixed by $[n]$ and $[m]$. In $(\ )_{\mathrm{A}}$ one keeps only the 'allowed' terms $[\lambda]$ defined in the following way:

(i) $\Sigma_{i=1}^{p}|l_i|\leqslant N$ ($l_i=\lambda_i$ T-type; $l_i=\lambda_i+\frac{1}{2}$ S-type), and $|l_i|\leqslant n_1$;

(ii) if l_1 or $|l_p|$ is equal to n_1, $[\lambda]$ must not contain any label λ_j such that $|l_j|\geqslant n_2$. If one of the l_j satisfies $|l_j|=n_2$, then $[\lambda]$ must not contain any label λ_k such that $|l_k|\geqslant n_3$, and so on;

(iii) if $n_i=l_i$, or $n_i=|l_{p-i+1}|$, $i=1,2,\ldots,p$ one must have $\Pi_{i=1}^{p} n_i=\Pi_{i=1}^{p} l_i$.

Moreover (a) an allowed GYT which appears more than once in a product of $[\nu]$ with $L_p^{2k}(\{\alpha_i\})$, for a fixed set $\{\alpha_i\}$, has to be considered only once, and (b) an allowed GYT which appears twice in the product of $[\nu]$ with L_p^{2k} with different sets $\{\alpha_i\}$ (this is possible only if $L_p^{2k}(\{\alpha_i\})$ has two more negative rows than $L_p^{2k}(\{\alpha'_i\})$) has to be considered twice if and only if

$$\sum_{i=1}^{p}|l_i|\leqslant N-2(k-1).$$

Finally, in the product $(\times)_{\mathrm{NA}}$ one will keep the 'non-allowed' GYT and the allowed GYT which should be neglected according to the above rules (a) and (b).

The last prescriptions are:

(i) in the final result one will keep only the GYT $[\lambda]$ which can be associated with an SO($2p$) IR which is such that $l_1\geqslant l_2\geqslant\ldots\geqslant|l_p|$;

(ii) IR which appear in the second term and not in the first term of the right-hand side of equation (4.2) have to be omitted.

If $a=1$, in Σ_3 of equation (4.2) one has to add, when it exists, the following expression

$$\sum_{k=1+b}^{Q}\sum_{r=0}^{k-b-1}(-1)^r[\mu+\tilde{L}_p^{2(k-2)}]\times(L_p^{2r}\times[\nu]) \tag{4.2a}$$

where $[\mu+\tilde{L}_p^{2(k-r)}]$ is a GYT whose rows are the rows of $[\mu]$ plus the rows of a negative GYT $\tilde{L}_p^{2(k-r)}(\{\gamma_i\})$ which can be constructed by the following recurrence formula ($k-r>2$)

$$\tilde{L}_p^{2(k-r)}=\sum_{\{\alpha_i\}}L_p^{2(k-r)}(\{\alpha_i\})-\sum_{\{\beta_i\}}(L_p^2\times L_p^{2(k-r-1)}(\{\beta_i\}))$$
$$+\sum_{l}\sum_{\{\gamma_i\}}(-1)^{l-k-r}(L_p^{2l}(\{\gamma_i\})\times\tilde{L}_p^{2(k-r-l)}) \tag{4.2b}$$

$$\tilde{L}_p^4=[0,0,\ldots,-1,-1,-2].$$

The new terms given in (4.2*a*) are present only when the IR $[m]$ has, at least, as many null labels as the number of negative rows of $\tilde{L}_p^{2(k-r)}$. We give here the $\tilde{L}_p^{2(k-r)}$ GYT for $k-r=3, 4$

$$L_p^6=[0,\ldots,0,-1,-1,-1,-3]; \qquad [0,\ldots,0,-2,-2,-2]$$

$$L_p^8=[0,\ldots,0,-1,-1,-1,-1,-4]; \qquad [0,\ldots,0,-1,-2,-2,-3].$$

Equation (4.2) looks very complicated at first sight, but it simplifies in practice. Let us remark that the second term for subtraction does not generally appear when $m_p \neq 0$ and that it can appear only for

$$m_{p-1}+m_p<k \qquad n_p \geq 0.$$

In particular, very compact formulae can be given in the cases of two completely symmetric or two completely antisymmetric SO(2p) representations if $m \geq n$

SO(2p)

$$[m,0,\ldots,0]\otimes[n,0,\ldots,0]=\sum_{l=0}^{n}\sum_{k}[m+n-k-2l,k,0,\ldots,0] \qquad (4.3)$$

where k satisfies $m+n-2l\geq 2k\geq 0$ and $(n-l)\geq k$.

SO(4q)

$$[m,m,\ldots,m]\otimes[n,n,\ldots,n]=\sum_{\{k_i\}}[m+n-k_1,m+n-k_1,\ldots,m+n-k_q,m+n-k_q]$$

$$[m,m,\ldots,m]\otimes[n,n,\ldots,-n]$$
$$=\sum_{\{k_i\}}[m+n,m+n-k_1,m+n-k_1,\ldots,m+n-k_{q-1},m+n-k_{q-1},m-n]$$

SO(4q+2)

$$[m,m,\ldots,m]\otimes[n,n,\ldots,n]$$
$$=\sum_{\{k_i\}}[m+n,m+n-k_1,m+n-k_1,\ldots,m+n-k_q,m+n-k_q]$$

$$[m,m,\ldots,m]\otimes[n,n,\ldots,-n]$$
$$=\sum_{\{k_i\}}[m+n-k_1,m+n-k_1,\ldots,m+n-k_q,m+n-k_q,m-n]$$

with the set $\{k_i\}$ satisfying

$$2n\geq k_q\geq k_{q-1}\geq\ldots\geq k_1\geq 0.$$

In order to illustrate the method, let us calculate in SO(10) the Kronecker product: $[11111]\otimes[\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}]$ of respective dimensions 126 and 144. We shall operate on the spinorial representation, since $N=\frac{7}{2}<M=5$, and consider the product $[\mu]\otimes[\nu]$ with $\mu_i=1$ $(i=1,2,\ldots,5)$ and $\nu_1=1$, $\nu_2=\ldots=\nu_5=0$. Since the quantity $Q=\Sigma_{i=1}^4 n_i=3$, we shall have to consider the cases $k=0,1,2,3$.

$k=0$

$$[11111]\otimes[10000]=[21111].$$

$k=1$

$$[000-1-1]\otimes[10000]=[100-1-1]_A+[0000-1]_A.$$

Therefore

$$[11111]\otimes\{[100-1-1]+[0000-1]\}=[21100]+[11110].$$

In this case there are only allowed GYT.

$k=2$

$$[0-1-1-1-1]\otimes[10000]=[1-1-1-1-1]_{\mathrm{A}}+[00-1-1-1]_{\mathrm{A}}$$

$$[000-2-2]\otimes[10000]=[100-2-2]_{\mathrm{NA}}+[000-1-2]_{\mathrm{A}}.$$

Therefore,

$$[11111]\otimes\{[1-1-1-1-1]+[00-1-1-1]+[000-1-2]\}$$
$$=[20000]+[11000]+[1110-1].$$

$k=3$

$$[0-1-1-2-2]\otimes[10000]=[1-1-1-2-2]_{\mathrm{NA}}+[00-1-2-2]_{\mathrm{NA}}+[0-1-1-1-2]_{\mathrm{A}}.$$

Therefore,

$$[11111]\otimes[0-1-1-1-2]_{\mathrm{A}}=[1000-1].$$

In this case there is no contribution from the NA diagrams. The final result is therefore, after adding $\frac{1}{2}$ to each index of the obtained GYT:

$$\underset{126}{[11111]}\otimes\underset{144}{[\tfrac{3}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}]}$$
$$=\underset{5280}{[\tfrac{5}{2}\,\tfrac{3}{2}\,\tfrac{3}{2}\,\tfrac{3}{2}\,\tfrac{3}{2}]}+\underset{8800}{[\tfrac{5}{2}\,\tfrac{3}{2}\,\tfrac{3}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}]}+\underset{1440}{[\tfrac{3}{2}\,\tfrac{3}{2}\,\tfrac{3}{2}\,\tfrac{3}{2}\,\tfrac{1}{2}]}+\underset{720}{[\tfrac{5}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}]}$$
$$+\underset{560}{[\tfrac{3}{2}\,\tfrac{3}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}]}+\underset{\overline{1200}}{[\tfrac{3}{2}\,\tfrac{3}{2}\,\tfrac{3}{2}\,\tfrac{1}{2}\;-\tfrac{1}{2}]}+\underset{\overline{144}}{[\tfrac{3}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\;-\tfrac{1}{2}]}.$$

Let us finally consider one of the simplest products requiring the use of 'NA' terms in equation (4.2), i.e. the product of SO(8) representations

$$[1100]\otimes[2100]$$

(of dimension 28 and 160 respectively)

$k=0$

$$[1100]\otimes[2100]=[3200]+[3110]+[2210]+[2111]$$

$k=1$

$$L^2\times[1100]=[11-1-1]_{\mathrm{NA}}+[100-1]_{\mathrm{A}}+[0000]_{\mathrm{A}}$$

$$\{[100-1]+[0000]\}\times[2100]=[3000]+3[2100]+[211-1]+[1110]$$

$k=2$

$$L^4(2)\times[1100]=[11-2-2]_{\mathrm{NA}}+[10-1-2]_{\mathrm{NA}}+[00-1-1]_{\mathrm{NA}}$$

$$L^4(11)\times[1100]=[00-1-1]_{\mathrm{NA}}$$

$$[00-1-1]_{\mathrm{A}}\times[2100]=[1000].$$

The GYT [0, 0, −1, −1] has to be considered once due to prescription (*b*). Now, we have to add the 'NA' terms:

$k = 1$

$$L^2 \times [2100] = [211-1]_{NA} + [200-1]_A + [110-1]_A + [1000]_A$$

$$[211-1]_{NA} \times [1100] = [2100].$$

We see at once that for $k = 2$ the 'NA' term gives no contribution since [2, 1, −1, −1] has two antisymmetrical boxes while $(L^2 \times [1, 1, 0, 0])_A$ has at most one positive box. So we cannot get any GYT acceptable to prescription (i).

The final result is:

$$\underset{28}{[1100]} \times \underset{160}{[2100]}$$

$$= \underset{1400}{[3200]} + \underset{1296}{[3110]} + \underset{840}{[2210]} + \underset{224}{[2111]} + \underset{224}{[211-1]} + \underset{112}{[3000]}$$

$$+ \underset{2\times 160}{2[2100]} + \underset{56}{[1110]} + \underset{8}{[1000]}.$$

5. The weights of an SO(2*p*) representation

In this section we give an outline of the proof of the rules given in § 4. The proof comes from the use of the Gel'fand–Zeitlin (GZ) basis (Gel'fand *et al* 1963) for SO(*n*) groups.

The infinitesimal generators $J_{2i-1,2i}(i = 1, 2, \ldots, p)$† of SO(2*p*) form a set of commuting Hermitian operators which are generally denoted by H_i and it is therefore possible to characterise (partially) a vector of the Hilbert space for any IR of SO(2*p*) by its set of eigenvalues λ_i. Each possible set of λ_i can be considered as the components of a vector, usually called the weight vector in an *n*-dimensional Euclidean space. It will be clear from the context whether we are referring to a vector in the Hilbert representation space or to a vector in the weight space being used to label the Hilbert vectors. The GZ vector is an eigenvector of the generators $J_{2i-1,2i}(i = 2, \ldots, p)$ only if it is an eigenvector with the maximum possible eigenvalue of $J_{2i-3,2i-2}$ and in this case the eigenvalue is just m_i, the *i*th label of the IR. However, it is possible to diagonalise these generators and introduce vectors which are specified by a set of *p* integers or half integers (positive, null or negative) $\lambda_i(i = 1, 2, \ldots, p)$. (The complete set of weights for any IR of SO(*n*) can be derived from the greatest weight with the help of the Dynkin diagram (see for example, Slansky 1980). However, to our knowledge, there has never been written a compact and explicit form for the weights of any SO(2*p*) IR.)

$$\begin{aligned} &J_{2i-1,2i}|[\lambda]\rangle = \lambda_i|[\lambda]\rangle \\ &\lambda_i = m_i - l_1^i + \sum_{j=1}^{i-1} \tilde{A}_i^{i-j} \end{aligned} \qquad (5.1)$$

where m_i are the integer or half integer numbers specifying the SO(2*p*) IR, l_k^i are all

† Our operators J_{ij} defined in § 2 differ from the generators I_{ik} of Gel'fand *et al* (1963) by a factor $-i$. With our definition, we get Hermitian operators.

possible non-negative integer numbers such that

$$0 \leqslant l_k^i \leqslant m_i \qquad i=1,2,\ldots,n \qquad k=1,\ldots,n-i$$

$$l_k^i \geqslant l_{k+1}^i \qquad l_k^i = 0 \qquad \text{for } i+k>n \qquad l_0^n=0$$

$$m_i - l_k^i \geqslant m_{i+1} - l_{k-i}^{i+1},$$

$$A_j^{i-j} = \delta_{j1}\min(m_i - l_1^i, m_{i-1} - l_1^{i-1}) + k_j^{i-j} \qquad j=1,2,\ldots,i-1$$

$$k_j^{i-j} = m_{i-j} - l_{j+1}^{i-j} - \max(m_{i-j} - l_j^{i-j}, m_{i-j+1} - l_j^{i-j+1})$$

$$A_1^0 = k_1^0 = 0,$$

the tilde on top of $\tilde{A}_j^{i-j}$ means that we have to take any number in the following set of values

$$A_j^{i-j}, A_j^{i-j}-2, \ldots, A_j^{i-j} - 2k_j^{i-j}.$$

Notice that in equation (5.1) we have to choose all the possible combinations of $\tilde{A}_i^{i-j}$. So actually we have not only one value λ_i for any i but a set of values. Our notation is slightly ambiguous, but it avoids overloading the formulae. If $l_1^i = m_i$, one has to omit, in the expression for λ_{i+1}, the term $m_{i+1} - l_1^{i+1}$. To the weight vectors specified by the set of λ_i of equation (5.1) for all possible choices of l_j^i and $\tilde{A}_j^{i-j}$, one has to add the vectors specified by a set of λ'_i obtained in equation (5.1), changing the sign of an even number of λ_i in all the possible ways.

This pattern of eigenvalues is rather complicated, but there are general features which are worth emphasising.

(i) The λ_i are all integers or half integers depending on whether m_i are integers or half integers.

(ii) For any weight $[\lambda]$, the sum $\Sigma_{i=1}^p \lambda_i$ differs from $M = \Sigma_{i=1}^p m_i$ by $2k(k = 0, 1, \ldots, Q)$ where $Q = \Sigma_{i=1}^{2q} m_i$ for SO($4q$) or SO($4q+2$).

(iii) There is, in general, a degeneracy in the eigenvectors except for the one specified by the greatest weight (that is, $\lambda_i = m_i$); there is no degeneracy when all the labels m_i are equal or in the fundamental one [10 . . . 0].

(iv) The sum over all the weights of the ith components λ_i is zero for any fixed i.

It should be stressed that the weight $[\lambda]$ is not a complete set of labels since one has to know which IR the weight $[\lambda]$ belongs to and moreover there are, in general, several vectors in each IR labelled by the same set of $[\lambda]$. It is convenient to define subsets, each being constituted of vectors, the sum of whose components is

$$\Lambda_r = M - 2r \qquad r = 0, 1, \ldots, Q. \tag{5.2}$$

If we associate the $[m_i]$ with the subset $\Lambda_0 = M$, it is possible to obtain all the vectors belonging to a subset Λ_r by multiplying $[m_i]$ by the negative GYT introduced in § 4; the different subsets will then be different terms in the product of the GYT.

The direct product of two IR $[m_i] \otimes [n_i]$ can be computed as follows: find the greatest weight of the direct product, i.e. $m_i + n_i$; calculate all the weights belonging to the IR $[m_i + n_i]$; remove these weights from the set of all weights $(\lambda_i + \lambda'_i)$ where $\lambda_i(\lambda'_i)$ are all the weights belonging to the IR $[m_i]$ ($[n_i]$); find the greatest weight in the remaining set and so on. This method, which seems very cumbersome in principle, can be expressed in the form of a simple algorithm of § 4 by the introduction of the GYT.

The fact that we have to remove a set of weights which belongs to the IR already determined is taken into account by acting with $L_p^{2k}(\{\nu_i\})$ on the smallest IR and with the help of the subtraction term in the left-hand side of equation (4.2).

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References

Barbieri R 1980 *Lectures given at the International School 'E Fermi', Varenna, Italy CERN preprint* TH-2935
Butler P H and Wybourne B G 1969 *J. Physique* **30** 655
Dehuai L, Wybourne B G and King R C 1981 *Symmetrized powers of rotation group representations, Mathematics Department Preprint,* University of Southampton
Ellis J 1980 *Lecture given at the 21st Scottish Universities Summer School in Physics, St. Andrews, Scotland CERN preprint* TH-2942
Fischler M 1980 *Fermilab preprint* PUB-80/49-THY
Gel'fand I M, Minlos R A and Shapiro Z Ya 1963 *Representations of Rotation and Lorentz Groups* (New York: Pergamon) p 353
Gell-Mann M 1980 in *Proc. Einstein Centennial Symposium on Gauge Theories and Unification of Physical Forces, Jerusalem 1979* ed Y Neeman (New York: Addison-Wesley)
Gell-Mann M, Ramond P and Slansky R 1978 *Rev. Mod. Phys.* **50** 721
King R C 1975a in *Proc. Conf. Group Theoretical Methods in Physics, Nijmegen* ed A Janner, J Janssen and M Boon (Berlin: Springer)
—— 1975b *J. Phys. A: Math. Gen.* **8** 429
Lichtenberg D B 1970 *Unitary Symmetry and Elementary Particles* (New York: Academic)
Littlewood D E 1950 *The Theory of Group Characters* 2nd edn (Oxford: Clarendon)
Murnaghan F D 1938 *The Theory of Group Representations* (Baltimore: Johns Hopkins)
Nanopoulos D V 1980 in *Proc. 15th Rencontre de Moriond* ed Tran Van Than
Slansky R 1980 *Los Alamos preprint* LAUR 80-3495 *Phys. Rep.* to appear